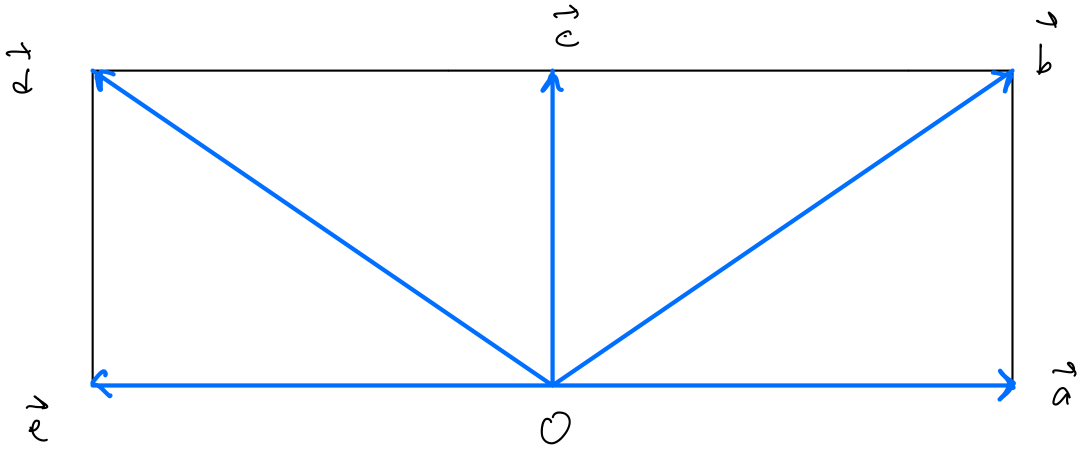
2/9/26

Last time we drew the picture



then

- $\vec{a} + \vec{c} = \vec{b}$
- $\vec{a} + \vec{d} = \vec{c}$
- $\vec{a} + \vec{e} = \vec{0}$
- $\vec{b} + \vec{d} = 2\vec{c}$

The linear transformations studied in §2.2 are related by these facts. They're useful for calculations. (See an example in a moment.)

The list of relations is given at the end of my lecture notes from Friday, although I didn't write it on the board, and a similar list can be found in the book just before Figure 7 in §2.2. (My list might be more useful.)

<u>Ex</u> Let ℓ be the line in $\mathbb{R}^3$ spanned by $\vec{0}$ and $\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$.

(a) Find the matrix giving projection to ℓ.

First we need to find our unit vector.

let $w = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$. Note $\|\vec{w}\|^2 = 9 + 16 + 25 = 50$ so $\|\vec{w}\| = \sqrt{50}$

$\vec{u} = \begin{bmatrix} 3/\sqrt{50} \\ 4/\sqrt{50} \\ 5/\sqrt{50} \end{bmatrix}$ $\quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\text{proj}_\ell(\vec{x}) = (\vec{x} \cdot \vec{u})\,\vec{u}$$

$$= \underbrace{\left(\frac{3}{\sqrt{50}} x_1 + \frac{4}{\sqrt{50}} x_2 + \frac{5}{\sqrt{50}} x_3 \right)}_{\text{scalar}} \overbrace{\begin{bmatrix} 3/\sqrt{50} \\ 4/\sqrt{50} \\ 5/\sqrt{50} \end{bmatrix}}^{\text{vector}}$$

$$= \begin{bmatrix} \frac{9}{50} x_1 + \frac{12}{50} x_2 + \frac{15}{50} x_3 \\ \frac{12}{50} x_1 + \frac{16}{50} x_2 + \frac{20}{50} x_3 \\ \frac{15}{50} x_1 + \frac{20}{50} x_2 + \frac{25}{50} x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 9/50 & 12/50 & 15/50 \\ 12/50 & \frac{16}{50} & 20/50 \\ 15/50 & \frac{20}{50} & \frac{25}{50} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(b) Find the matrix giving reflection about ℓ.

$$\boxed{\text{ref}_\ell(\vec{x}) = 2\,\text{proj}_\ell(\vec{x}) - \vec{x}}$$

So the matrix is

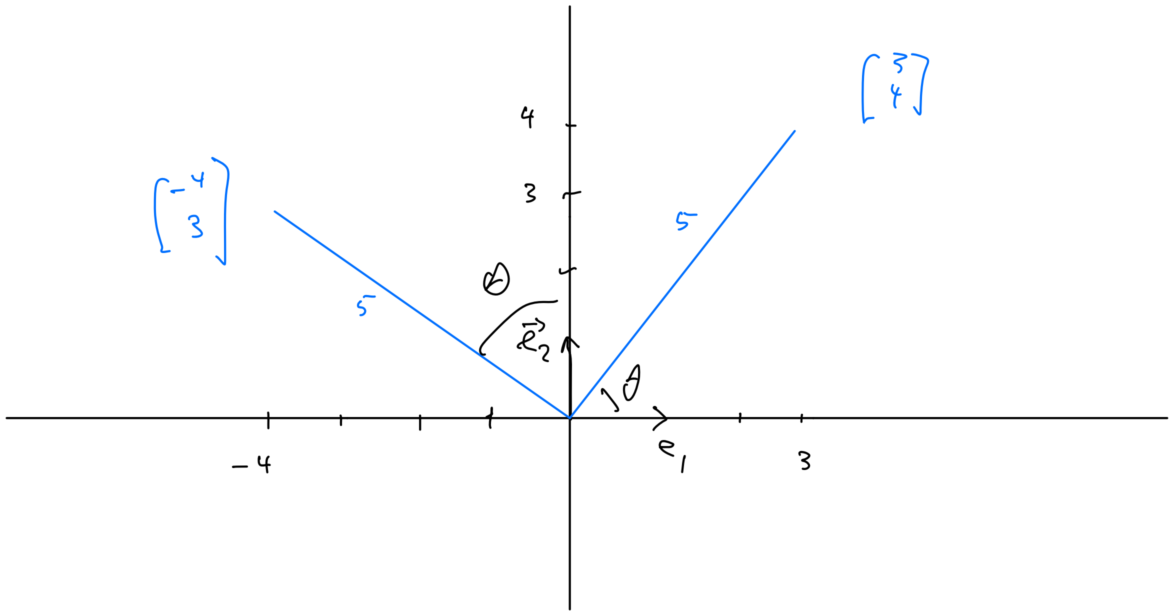
$$2 \begin{bmatrix} 9/50 & 12/50 & 15/50 \\ 12/50 & 16/50 & 20/50 \\ 15/50 & 20/50 & 25/50 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{18}{50} & \frac{24}{50} & \frac{30}{50} \\ \frac{24}{50} & \frac{32}{50} & \frac{40}{50} \\ \frac{30}{50} & \frac{40}{50} & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -32/50 & 24/50 & 30/50 \\ 24/50 & -18/50 & 40/50 \\ 30/50 & 40/50 & 0 \end{bmatrix}$$

We also talked about rotations.

Ex Let $A = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$ Describe the geometric effects of the corresp. linear transformation

Since $A = \left[T(\vec{e}_1) \mid T(\vec{e}_2) \right]$, let's study these columns geometrically



$\left\| \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -4 \\ 3 \end{bmatrix} \right\| = 5$

so A combines scaling by 5 and rotating by $\theta = \arccos\left(\frac{3}{5}\right)$

like the above example

You should read Th. 2.2.4, 2.2.5

shears

Give idea of shears using cards. Matrix looks like $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

horiz shear   vert. shear

See useful list of matrices right after Th 2.2.5

§2.3 Matrix Products

Has to do with the composition of two linear transformations.

First let's formally say what a matrix product is. (Def 2.3.1)
(We've already seen the main step.) Just do it by example. (But see Th 2.3.4)

Recall: if f, g are functions then $(f \circ g)(x) = f(g(x))$

On the other hand, matrix products are defined by a series of dot products, as long as they're defined

Running Ex

$$\underset{(4\times 2)}{\overset{B}{\begin{bmatrix} 0 & 1 \\ 2 & 3 \\ -1 & 2 \\ 3 & 1 \end{bmatrix}}} \underset{(2\times 3)}{\overset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 1 \end{bmatrix}}} = \underset{(4\times 3)}{\begin{bmatrix} 3 & 4 & 1 \\ 11 & 16 & 9 \\ 5 & 6 & -1 \\ 6 & 10 & 10 \end{bmatrix}} \leftarrow BA$$

But if we add a row to A (for example), it's no longer defined.

What does this have to do with lin. transformations or with composition of functions?